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LOCALLY CONVEX FUNCTIONS AND THE SCHWARZIAN DERIVATIVE

BY
REUVEN HARMELIN
Department of Mathematics, Technion — Israel Institute of Technology, Haifa, 32000 Israel

ABSTRACT
In this paper we study various classes of locally convex analytic functions in the unit disc, which are invariant under the group of Möbius automorphisms of the unit disc. Bounds for the Schwarzian derivative of functions in these classes are achieved and used to obtain estimates for the uniform hyperbolic radius of univalence in these classes.

1. Introduction

Let $f(z)$ be a locally univalent analytic function in the unit disc $\Delta = \{z : |z| < 1\}$. Then $f(z)$ is convex in Δ (i.e. $D = f(\Delta)$ is convex in $\mathbf{C}$) if and only if $f(z)$ satisfies either one of the following two inequalities (see [1] p. 5):

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0, \qquad z \in \Delta \tag{1.1}$$

or

$$\left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leqq 2, \qquad z \in \Delta. \tag{1.2}$$

On the other hand Nehari proved in [6] that if $f(z)$ is convex in Δ, then

$$(1 - |z|^2)^2 |S_f(z)| \leqq 2, \qquad z \in \Delta, \tag{1.3}$$

where $S_f(z)$ is the Schwarzian derivative of $f(z)$, defined by

$$S_f(z) = \varphi_f'(z) - \tfrac{1}{2}\varphi_f(z)^2, \qquad \varphi_f(z) = f''(z)/f'(z).$$

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In view of inequalities (1.1), (1.2) and (1.3) we define the following three classes of analytic functions in Δ:

(i) Let $K(r)$ be the class of all locally univalent analytic functions in Δ which are convex on every hyperbolic disc in Δ of hyperbolic radius $\rho = \frac{1}{2}\log((1+r)/(1-r))$ for some $r \leqq 1$. In other words $f \in K(r)$ if and only if $f \circ g$ satisfies inequality (1.1) in the disc $\Delta_r = \{z : |z| < r\}$ for every $g(z) = (z+\zeta)/(1+\bar{\zeta}z)$, $\zeta \in \Delta$.

(ii) Let $\mathscr{F}(\eta)$ be the class of all locally univalent analytic functions in Δ which satisfy the inequality

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leqq 2\eta \tag{1.2'}$$

for all $z \in \Delta$ and some $\eta \geqq 1$.

(iii) Let $\mathscr{S}(\beta)$ be the class of all locally univalent analytic functions in Δ which satisfy the inequality

$$(1-|z|^2)^2 |S_f(z)| \leqq 2\beta \tag{1.3'}$$

for all $z \in \Delta$ and some $\beta \geqq 0$.

Thus, Nehari's result [6] may be stated as follows:

$$\mathscr{F}(1) \subset \mathscr{S}(1).$$

Also, for $\eta > 1$ we have by Satz 2.4 in [7]:

$$\mathscr{F}(\eta) \subset \mathscr{S}(\eta^2 + 3\sqrt{3}\,\eta + 3).$$

Other relations between the classes $\mathscr{F}(\eta)$, $K(r)$ and $\mathscr{S}(\beta)$ which have been discovered by Pommerenke in [7] are:

$$\mathscr{F}(\eta) \subset K(\eta - \sqrt{\eta^2 - 1}) \qquad \text{(Satz 2.5 in [7])} \tag{1.4}$$

and

$$\mathscr{S}(\beta) \subset \mathscr{F}(\sqrt{1+\beta}) \qquad \text{(Folgerung 2.3 in [7]).} \tag{1.5}$$

In this paper we show that

$$K(r) \subset \mathscr{F}(1/r) \cap \mathscr{S}(1/r^2) \qquad \text{(see Theorem 2)} \tag{1.6}$$

and

$$\mathscr{F}(\eta) \subset \mathscr{S}(\beta(\eta)) \qquad \text{(see Theorem 3)} \tag{1.7}$$

where $\beta(1)=1$ and $1<\beta(\eta)\leqq 1+\eta^2$ for $\eta>1$, so that (1.7) is an improvement on Satz 2.4 in [7].

Both (1.6) and (1.7) follow as consequences of Theorem 1 in which we consider a family of classes of locally convex analytic functions in Δ, and for each given class in this family we find values for η and β such that the given class is contained in $\mathscr{F}(\eta)\cap\mathscr{S}(\beta)$.

Using results of Beesack–Schwarz [2] and Minda [4] we derive from (1.6) and (1.7) estimates for the uniform radius of univalence of functions in $K(r)$ and $\mathscr{F}(\eta)$. In particular we obtain relations between the uniform radii of convexity and univalence for every universal covering map from Δ onto any hyperbolic domain of any connectivity.

In the last section we apply the same technique as in the proof of Theorem 1 to obtain a refinement of Nehari's result (1.3) in the class of convex functions of order α in Δ (see [3]) for all $\alpha\in(0,1)$.

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2. Ω-Local convexity

Let $f(z)$ be a locally univalent analytic function in Δ. Denote $\varphi_f(z)=f''(z)/f'(z)$ and $\varphi_f(z;\zeta)=\varphi_{f\circ g}(z)=(f\circ g)''(z)/(f\circ g)'(z)$, where $g(z)=(z+\zeta)/(1+\bar{\zeta}z)$, $\zeta\in\Delta$. Then

$$(2.1)\quad (1+\bar{\zeta}z)^2\varphi_f(z;\zeta)=(1-|\zeta|^2)\varphi_f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right)-2\bar{\zeta}(1+\bar{\zeta}z),\quad (z,\zeta)\in\Delta\times\Delta.$$

Let Ω be a simply connected domain in $\mathbf{C}$ with at least two boundary points and $r\in(0,1)$. If the condition

$$(2.2)\quad w=1+z\varphi_f(z;\zeta)\in\Omega,\quad \text{for every } z\in\Delta_r=\{z:|z|<r\}\text{ and all }\zeta\in\Delta.$$

holds, we say that $f(z)$ is Ω-locally convex with radius r, and denote by $K(\Omega;r)$ the class of all such functions.

REMARKS. (i) Obviously by definition we have

$$(2.3)\qquad K(r)=K(\Omega;r)\qquad\text{for }\Omega=\{w:\operatorname{Re}w>0\}.$$

(ii) Every class $K(\Omega;r)$ is linearly-invariant in the sense that $f\circ g\in K(\Omega;r)$ for every $f\in K(\Omega;r)$ and $g(z)=(z+\zeta)/(1+\bar{\zeta}z)$, $(z,\zeta)\in\Delta\times\Delta$.

(iii) By condition (2.2) we notice that $1\in\Omega$. Hence, by the Riemann

Mapping Theorem there exists a unique analytic function $\gamma(w)$ that maps Ω onto Δ, such that $\gamma(1)=0$ and $\gamma'(1)>0$.

THEOREM 1. *If $f \in K(\Omega; r)$ for a given Ω and $r \in (0,1]$, then*

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right| \leqq \gamma'(1)^{-1}r^{-1}, \qquad z\in\Delta \tag{2.4}$$

and

$$(1-|z|^2)^2|S_f(z)| + C\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right|^2 \leqq \gamma'(1)^{-1}r^{-2}, \qquad z\in\Delta, \tag{2.5}$$

where $C=\gamma'(1)-\frac{1}{2}|1+\gamma''(1)/\gamma'(1)|$.

PROOF. By definitions of $K(\Omega; r)$ and $\gamma(w)$, it readily follows that the composition $\gamma(1+z\varphi_f(z;\zeta))$ is an analytic function of z which maps $\Delta_r = \{z : |z| < r\}$ into Δ and it vanishes at $z=0$. Hence by the Schwarz Lemma we conclude that the function

$$F(z;\zeta)=\gamma(1+z\varphi_f(z;\zeta))/z, \qquad (z,\zeta)\in\Delta\times\Delta$$

is also analytic in z and maps Δ_r into $\Delta_{r^{-1}}$. Theorefore we have in particular

$$|F(0;\zeta)| \leqq 1/r, \qquad \zeta\in\Delta, \tag{2.6}$$

and by the invariant formulation of the Schwarz Lemma we obtain

$$\frac{r|F'_z(z;\zeta)|}{1-r^2|F(z;\zeta)|^2} \leqq \frac{r}{r^2-|z|^2}, \qquad (z,\zeta)\in\Delta_r\times\Delta.$$

Hence we derive at $z=0$:

$$|F'_z(0;\zeta)| + |F(0;\zeta)|^2 \leqq 1/r^2, \qquad \zeta\in\Delta. \tag{2.7}$$

Expanding $F(z;\zeta)$ into a power series in z:

$$F(z;\zeta)=\gamma'(1)\varphi_f(0;\zeta)+\left\{\gamma'(1)\frac{\partial}{\partial z}\varphi_f(z;\zeta)\bigg|_{z=0}+\tfrac{1}{2}\gamma''(1)\varphi_f(0;\zeta)^2\right\}z+\cdots$$

and using (2.1), we get

$$F(0;\zeta)=\gamma'(1)\varphi_f(0;\zeta)=\gamma'(1)[(1-|\zeta|^2)\varphi_f(\zeta)-2\bar{\zeta}]$$

and

$$\gamma'(1)^{-1}F_z'(0;\zeta)=\frac{\partial}{\partial z}\varphi_f(z;\zeta)\Big|_{z=0}+\frac{1}{2}\frac{\gamma''(1)}{\gamma'(1)}\varphi_f(0;\zeta)^2$$

$$=(1-|\zeta|^2)^2\varphi_f'(\zeta)-2\bar{\zeta}[(1-|\zeta|^2)\varphi_f(\zeta)-\bar{\zeta}]+\frac{1}{2}\frac{\gamma''(1)}{\gamma'(1)}\varphi_f(0;\zeta)^2$$

$$=(1-|\zeta|^2)^2[\varphi_f'(\zeta)-\tfrac{1}{2}\varphi_f(\zeta)^2]+\frac{1}{2}\left(1+\frac{\gamma''(1)}{\gamma'(1)}\right)[(1-|\zeta|^2)\varphi_f(\zeta)-2\bar{\zeta}]^2.$$

Thus, inequalities (2.4) and (2.5), respectively, readily follow from (2.6) and (2.7). q.e.d.

COROLLARY 1. *Given* Ω *and* $r\in(0,1)$ *as above, then*

$$K(\Omega;r)\subset\mathscr{F}(\eta)\cap\mathscr{S}(\beta) \tag{2.8}$$

where

$$\eta=\eta(\Omega;r)=\tfrac{1}{2}\gamma'(1)^{-1}r^{-1} \tag{2.9}$$

and

$$\beta=\beta(\Omega;r)=\tfrac{1}{2}\max\{\gamma'(1),\tfrac{1}{2}|1+\gamma''(1)/\gamma'(1)|\}\gamma'(1)^{-2}r^{-2}. \tag{2.10}$$

PROOF. By (2.4) we see that $K(\Omega;r)$ lies in $\mathscr{F}(\eta)$, for the η given by (2.9). Also, if $\gamma'(1)\geqq\frac{1}{2}|1+\gamma''(1)/\gamma'(1)|$, then $C\geqq 0$ and hence we have by (2.5)

$$(1-|z|^2)^2|S_f(z)|\leqq\gamma'(1)^{-1}r^{-2}=2\beta(\Omega;r)$$

for every $f\in K(\Omega;r)$ and all $z\in\Delta$.

Finally, if $\gamma'(1)<\frac{1}{2}|1+\gamma''(1)/\gamma'(1)|$, so that $C<0$, then (2.5) and (2.4) imply

$$(1-|z|^2)^2|S_f(z)|\leqq\frac{1}{2}\left|1+\frac{\gamma''(1)}{\gamma'(1)}\right|\gamma'(1)^{-2}r^{-2}=2\beta(\Omega;r)$$

for all $f\in K(\Omega;r)$ and $z\in\Delta$. q.e.d.

REMARK. Using Satz 2.5 in [7] (see (1.4)) and a generalization of Nehari's univalence criterion [5] due to Beesack–Schwarz [2] and Minda [4], one may derive from Corollary 1 lower bounds for the uniform hyperbolic radii of convexity and univalence for all f in $K(\Omega;r)$.

In another direction one can localize (2.4) and (2.7) and derive coefficient inequalities for normalized functions in $K(\Omega;r)$:

COROLLARY 2. *If $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \in K(\Omega; r)$ for given Ω and $r \in (0, 1]$, then*

$$|a_2| \leqq \tfrac{1}{2}\gamma'(1)^{-1} r^{-1} \tag{2.11}$$

and

$$|a_3| \leqq \tfrac{1}{6}\max\left\{\gamma'(1), \left|1 - \frac{1}{2}\frac{\gamma''(1)}{\gamma'(1)}\right|\right\}\gamma'(1)^{-2} r^{-2}. \tag{2.12}$$

Next, applying Theorem 1 to the class $K(\delta, r) = K(\Omega(\delta); r)$, where $\Omega(\delta) = \{w : |\arg w| < \pi\delta/2\}$, for $0 \leqq r \leqq \min(1, \delta)$, $\delta > 0$, we obtain the relation (1.6) in an improved form:

THEOREM 2. *If $f \in K(\delta, r)$ for given δ, r as above, then*

$$(1 - |z|^2)^2 |S_f(z)| + \frac{1}{2\delta}\left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right|^2 \leqq 2\delta r^{-2}, \quad z \in \Delta. \tag{2.13}$$

Hence we have in particular

$$K(\delta, r) \subset \mathscr{F}(\delta/r) \cap \mathscr{S}(\delta/r^2). \tag{2.14}$$

PROOF. Indeed, in this case we take $\gamma(w) = (w^{1/\delta} - 1)/(w^{1/\delta} + 1)$ which maps $\Omega(\delta)$ onto Δ and compute:

$$\gamma'(1) = 1/2\delta, \quad \frac{\gamma''(1)}{\gamma'(1)} = -1 \quad \text{and} \quad C = \gamma'(1) = 1/2\delta.$$

Thus (2.5) readily yields (2.13), and (2.14) as well. q.e.d.

Notice that in the case $r = \delta = 1$, Theorem 2 implies an improvement on Nehari's necessary condition (1.3) for convexity (cf. [8]):

COROLLARY 3. *Let $f(z)$ be a convex analytic function in Δ. Then*

$$(1 - |z|^2)^2 |S_f(z)| + \frac{1}{2}\left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right|^2 \leqq 2 \tag{2.13$'$}$$

holds for all $z \in \Delta$ and is sharp, as equality is attained identically in Δ by all the functions

$$f_\beta(z) = \{(1 - z)/(1 + z)\}^\beta, \qquad \text{for } \beta \in (-1, 1), \quad \beta \neq 0$$

and

$$f_0(z) = \log[(1 - z)/(1 + z)].$$

Theorem 3 in [2] (or Theorem 3 in [4]) yields the following geometric interpretation of Theorem 2 in the case $\delta = 1$:

COROLLARY 4. *If $f \in K(r)$ for some $r \in (0, 1)$, then $f(z)$ is uniformly locally univalent in the sense of* [4], *that is, $f(z)$ is univalent in every hyperbolic disc in Δ, of hyperbolic radius*

$$\rho_u(f) \geqq \frac{\pi}{2} \frac{r}{\sqrt{1 - r^2}} = \frac{\pi}{2} \sinh \rho_c(f) \tag{2.15}$$

where $\rho_c(f) = \frac{1}{2}\log[(1 - r)/(1 + r)]$ is the uniform hyperbolic radius of convexity of $f(z)$.

PROOF. By Theorem 3 in [2] (cf. [4]), $f(z)$ is univalent in every hyperbolic disc in Δ of hyperbolic radius $\rho = \pi/2k$, provided that $f \in \mathscr{S}(1 + k^2)$. On the other hand, by Theorem 2, if $f \in K(r)$, then $f \in \mathscr{S}(r^{-2})$, i.e. $k = (r^{-2} - 1)^{1/2}$, and (2.15) follows. q.e.d.

REMARK. Conversely, from Theorem 4 in [4], Folgerung 2.3 and Satz 2.5 in [7] we conclude that if $f(z)$ is univalent in every hyperbolic disc in Δ of a pseudo-hyperbolic radius $R = \tanh \rho_u(f) = r_u(f) > 0$, then

$$f \in \mathscr{S}(3R^{-2}) \subset \mathscr{F}(\sqrt{1 + 3R^{-2}}) \subset K(\sqrt{1 + 3R^{-2}} - \sqrt{3}R^{-1}).$$

3. The class $\mathscr{F}(\eta)$ and the Schwarzian derivative

In this section we apply Theorem 1 to the class $\mathscr{F}(\eta)$ and find, in particular, an estimate for β, such that $\mathscr{F}(\eta) \subset \mathscr{S}(\beta)$.

First we show that every function $f \in \mathscr{F}(\eta)$ is $\Omega(\eta, r)$-locally convex with radius r, for all $r \in (0, 1)$, where

$$\Omega(\eta, r) = \left\{ w : \left| w - \frac{1 + r^2}{1 - r^2} \right| < \frac{2\eta r}{1 - r^2} \right\}.$$

LEMMA 1. *Let $\eta \geqq 1$. Then*

$$\mathscr{F}(\eta) = \bigcap \{K(\Omega(\eta, r); r), r \in (0, 1)\}. \tag{3.1}$$

PROOF. Observe, first, that condition (1.2′) may be written as follows:

$$\left| \frac{(f \circ g)''(0)}{(f \circ g)'(0)} \right| \leqq 2\eta \tag{3.2}$$

for every Möbius automorphism $g(\zeta) = (\alpha\zeta + z)/(1 + \alpha\bar{z}\zeta)$ of Δ. Hence, the class $\mathscr{F}(\eta)$ is linearly-invariant (cf. [7]) in the sense that $f \circ g \in \mathscr{F}(\eta)$ for every $f \in \mathscr{F}(\eta)$ and all Möbius automorphisms $g(\zeta)$ of Δ. Also notice that inequality (1.2′) may be written in the form

$$\left|1 + z\varphi_f(z) - \frac{1+|z|^2}{1-|z|^2}\right| \leqq \frac{2\eta|z|}{1-|z|^2}, \qquad \varphi_f(z) = f''(z)/f'(z), \quad z \in \Delta. \tag{3.3}$$

Hence, $f \in \mathscr{F}(\eta)$ if and only if

$$\left|1 + z\varphi_f(z;\zeta) - \frac{1+|z|^2}{1-|z|^2}\right| \leqq \frac{2\eta|z|}{1-|z|^2},$$

holds for all $(z,\zeta) \in \Delta \times \Delta$, where $\varphi_f(z;\zeta)$ is defined in (2.1). This means that $w = 1 + z\varphi_f(z;\zeta) \in \overline{\Omega(\eta,|z|)}$ for all $(z,\zeta) \in \Delta \times \Delta$. But $\overline{\Omega(\eta,|z|)} \subset \Omega(\eta,r)$ whenever $|z| < r$, for every $r \leqq 1 \leqq \eta$ and this completes the proof. q.e.d.

Theorem 3. *If $f \in \mathscr{F}(\eta)$ for some $\eta \geqq 1$, then*

$$(1-|z|^2)^2|S_f(z)| + \frac{1}{2}\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right|^2 \leqq 2p(\eta), \quad z \in \Delta \tag{3.4}$$

and

$$(1-|z|^2)^2|S_f(z)| \leqq 2\beta(\eta), \qquad z \in \Delta, \tag{3.5}$$

where

$$p(\eta) = \eta^2 + (\eta + 1/2\eta)\sqrt{\eta^2-1}.$$

and $1 \leqq \beta(\eta) \leqq 1 + \eta^2$ is given by:

$$\beta(\eta) = 1 + \eta^2 \qquad \text{for } \eta \geqq \sqrt{1+\sqrt{2}} \tag{3.6}$$

and

$$\beta(\eta) = \left\{\frac{1}{8\eta^2}[(27\eta^4 - 18\eta^2 - 1) + (\eta^2-1)^{1/2}(9\eta^2-1)^{3/2}]\right\}^{1/2} \qquad \text{for } 1 \leqq \eta \leqq \sqrt{1+\sqrt{2}}. \tag{3.6′}$$

Proof. By Lemma 1, if $f \in \mathscr{F}(\eta)$ then $f \in K(\Omega(\eta,r);r)$ for every $r \in (0,1)$. Therefore we can apply Theorem 1 to the case that $\Omega = \Omega(\eta,r)$ for a fixed $\eta \geqq 1$ and a variable $r \in (0,1)$. Thus

$$\gamma(w) = \frac{\eta(1-r^2)(w-1)}{r[(1-r^2)w + (2\eta^2 - r^2 - 1)]} : \Omega(\eta,r) \to \Delta,$$

$$\gamma'(1) = \frac{\eta(1-r^2)}{2r(\eta^2-r^2)}, \qquad \frac{\gamma''(1)}{\gamma'(1)} = -\frac{1-r^2}{\eta^2-r^2}$$

and

$$C = \gamma'(1) - \tfrac{1}{2}|1 + \gamma''(1)/\gamma'(1)| = C(r) = \frac{1-\eta r}{2r(\eta - r)}.$$

Hence inequality (2.5) yields:

$$(3.7)\quad (1-|z|^2)^2|S_f(z)| + C(r)\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right|^2 \leqq 2q(r), \quad z \in \Delta,$$

where

$$q(r) = \tfrac{1}{2}\gamma'(1)^{-1}r^{-2} = (\eta^2 - r^2)/[\eta r(1-r^2)].$$

Now if we set $r = \eta - \sqrt{\eta^2 - 1}$, (3.7) implies (3.4) with the given $p(\eta)$.

Next, notice that $C(r) > 0$ only for $r \in (0, 1/\eta)$ and therefore

$$(3.7')\qquad (1-|z|^2)^2|S_f(z)| \leqq \begin{cases} 2q(r) & \text{for } 1 \leqq r \leqq 1/\eta, \\ 2q(r) - 4\eta^2 C(r) & \text{for } r \geqq 1/\eta. \end{cases}$$

Observe now that the minimum value of $q(r)$ in the interval (0, 1) is attained at $r_0 = \{\tfrac{1}{2}[(3\eta^2 - 1) - (\eta^2-1)^{1/2}(9\eta^2-1)^{1/2}]\}^{1/2}$, and $q(r_0) = \beta(\eta)$ as given in (3.6′). But $r_0 \leqq 1/\eta$ only for $\eta \leqq \sqrt{1+\sqrt{2}}$, and if $\eta \geqq \sqrt{1+\sqrt{2}}$ then $q(r) \geqq q(1/\eta) = 1 + \eta^2$ for all $r \in (0, 1/\eta]$. On the other hand

$$q(r) - 2\eta^2 C(r) = \frac{\eta^2 - 1}{\eta}\left(\frac{\eta^2}{\eta - r} + \frac{r}{1-r^2}\right)$$

which obviously is an increasing function of r in the interval $1/\eta \leqq r < 1$, and therefore we get in that interval

$$q(r) - 2\eta^2 C(r) \geqq q(1/\eta) - 2\eta^2 C(1/\eta) = q(1/\eta) = 1 + \eta^2.$$

This proves (3.5) with the $\beta(\eta)$ given by (3.6) and (3.6′). q.e.d.

Using the arguments of the proof of Corollary 4 we conclude:

Corollary 5. *Let $f \in \mathscr{F}(\eta)$ for some $\eta \geqq 1$. Then $f(z)$ is univalent in every hyperbolic disc in Δ with the hyperbolic radius*

$$(3.8)\qquad \rho_u(f) \geqq \pi/2\sqrt{\beta(\eta) - 1} \geqq \pi/2\eta.$$

Next we derive some coefficient inequalities for normalized functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ in $\mathscr{F}(\eta)$.

COROLLARY 6. *Let* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n \in \mathscr{F}(\eta)$ *for some* $\eta \geqq 1$. *Then*

$$|a_3-a_2^2|+\tfrac{1}{3}|a_2|^2 \leqq \tfrac{1}{3}[\eta^2+(\eta+1/2\eta)\sqrt{\eta^2-1}], \tag{3.9}$$

$$|a_3-a_2^2| \leqq \beta(\eta)/3 \leqq (1+\eta^2)/3 \tag{3.10}$$

and

$$|a_3| \leqq \eta^2-\frac{2\eta^2-1}{6\eta}(2\eta-\sqrt{\eta^2+3}). \tag{3.11}$$

PROOF. Inequalities (3.9) and (3.10) are localized versions of (3.4) and (3.5), respectively, at $z=0$.

Next, if we substitute the values of $\gamma'(1)$ and $\gamma''(1)/\gamma'(1)$, that have been computed in the proof of Theorem 3, into (2.12), we obtain

$$|a_3| \leqq \tfrac{1}{3}\max[(1-r^2)\eta,(1+2\eta^2-3r^2)r]\frac{\eta^2-r^2}{\eta^2 r(1-r^2)^2}=A_3(r),$$

for every $r\in(0,1)$, and hence

$$|a_3| \leqq \min_{0<r<1} A_3(r)=A_3((\sqrt{\eta^2+3}-\eta)/3)$$

$$=\eta^2-\frac{2\eta^2-1}{6\eta}(2\eta-\sqrt{\eta^2+3}). \qquad \text{q.e.d.}$$

4. Convex functions of order α

An analytic function $f(z)$ in Δ is convex of order α, for some $\alpha\in(0,1)$, if $f(z)$ satisfies the improved convexity condition

$$1+\operatorname{Re}\frac{zf''(z)}{f'(z)}>\alpha, \qquad z\in\Delta \quad (\text{see [3]}). \tag{4.1}$$

Although the class of all convex functions of order α in Δ is not a linearly-invariant family, for $\alpha>0$, the technique of the proof of Theorem 1 may be applied here to derive sharp bounds for $|(1-|z|^2)\varphi_f(z)-2\bar{z}|$ and for $(1-|z|^2)^2|S_f(z)|$.

THEOREM 4. *Let $f(z)$ be an analytic function in Δ. Then the following statements are equivalent*:

(i) *$f(z)$ is convex of order α in Δ.*

(ii)

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right| \leqq 2[1-\alpha(1-|z|^2)]^{1/2}, \qquad z\in\Delta. \tag{4.2}$$

(iii)

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right| \leqq 2[1-\alpha(1-|z|)], \qquad z\in\Delta. \tag{4.3}$$

(iv)

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2(1-\alpha)\bar{z}\right| \leqq 2(1-\alpha), \qquad z\in\Delta. \tag{4.4}$$

(v)

$$(1-|z|^2)^2|S_f(z)|+\frac{1}{2}\left|(1-|z|^2)\frac{f''(z)}{f'(z)}-2\bar{z}\right|^2 \leqq 2[1-\alpha(1-|z|^2)], \; z\in\Delta. \tag{4.5}$$

PROOF. The implications (v)$\Rightarrow$(ii) and (iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are trivial. Therefore we just have to show that (ii)$\Rightarrow$(i) and (i) implies both (iv) and (v). Indeed if we square (4.2) and simplify we obtain

$$0 \leqq (1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|^2 \leqq 4\left(1-\alpha+\operatorname{Re}\frac{zf''(z)}{f'(z)}\right)$$

and hence inequality (4.2) implies (4.1).

Conversely, inequality (4.1) tells us that the analytic function $w=1+z\varphi_f(z)$, $\varphi_f(z)=f''(z)/f'(z)$, maps the unit disc Δ into the half plane $\{w : \operatorname{Re} w>\alpha\}$. This half plane is mapped back onto Δ by the mapping $g(w)=(w-1)/(w+1-2\alpha)$. Hence the composition of these two functions:

$$g(1+z\varphi_f(z))=\frac{z\varphi_f(z)}{2(1-\alpha)+z\varphi_f(z)}, \qquad \varphi_f(z)=f''(z)/f'(z),$$

satisfies the requirements of the Schwarz Lemma, and therefore

$$F(z)=\frac{g(1+z\varphi_f(z))}{z}=\frac{\varphi_f(z)}{2(1-\alpha)+z\varphi_f(z)}$$

maps Δ into itself. Hence we have

$$|F(z)| \leqq 1 \qquad \text{for all } z\in\Delta, \tag{4.6}$$

and moreover, by the invariant formulation of the Schwarz Lemma [1, p. 3], we obtain

$$(4.7)\qquad (1-|z|^2)|F'(z)| \leqq 1-|F(z)|^2 \qquad \text{for all } z\in\Delta.$$

Inequality (4.6) may be written as follows:

$$|\varphi_f(z)|^2 \leqq 4(1-\alpha)^2 + 4(1-\alpha)\mathrm{Re}\, z\varphi_f(z) + |z\varphi_f(z)|^2,$$

or

$$(1-|z|^2)\left(|\varphi_f(z)|^2 - \frac{4(1-\alpha)}{(1-|z|^2)}\mathrm{Re}\, z\varphi_f(z) + \frac{4(1-\alpha)^2}{(1-|z|^2)^2}|z|^2\right)$$

$$\leqq 4(1-\alpha)^2\left(1+\frac{|z|^2}{1-|z|^2}\right)$$

which is inequality (4.4). Similarly, inequality (4.7) may be written in the form

$$(1-|z|^2)|2(1-\alpha)\varphi_f'(z)-\varphi_f(z)^2| \leqq |2(1-\alpha)+z\varphi_f(z)|^2 - |\varphi_f(z)|^2$$

and, since $0\leqq\alpha<1$, this inequality yields

$$(1-|z|^2)\{2(1-\alpha)|\varphi_f'(z)-\tfrac{1}{2}\varphi_f(z)^2| - \alpha|\varphi_f(z)|^2\}$$

$$\leqq 4(1-\alpha)[1-\alpha+\mathrm{Re}\, z\varphi_f(z)] - (1-|z|^2)|\varphi_f(z)|^2$$

and simplifying it we get

$$(1-|z|^2)\left\{|\varphi_f'(z)-\tfrac{1}{2}\varphi_f(z)^2| + \tfrac{1}{2}|\varphi_f(z)|^2 - \frac{2}{1-|z|^2}\mathrm{Re}\, z\varphi_f(z)\right\}$$

$$\leqq 2(1-\alpha), \quad z\in\Delta.$$

Finally, adding $2|z|^2/(1-|z|^2)$ to both sides, inequality (4.5) follows at once.

The functions $f(z)=(1-z)^{2\alpha-1}$ for $\alpha \neq \frac{1}{2}$, or $f(z)=\log(1-z)$ for $\alpha=\frac{1}{2}$, show that all the conditions (ii)–(v) are sharp as necessary conditions. Furthermore, these functions satisfy the equalities in (4.4) and (4.5) identically in Δ.

q.e.d.

Inequalities (4.4) and (4.5) readily yield at $z=0$:

Corollary 7. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be a convex function of order α in Δ, for some $\alpha \in (0,1)$. Then*

(4.8) $$|a_2| \leqq 1-\alpha, \qquad |a_3| \leqq \tfrac{1}{3}(1-\alpha)(3-2\alpha),$$

and equalities are attained in both inequalities by the coefficients of the function $f(z)=[1-(1-z)^{2\alpha-1}]/(2\alpha-1)$ for $\alpha \neq \frac{1}{2}$, or $f(z)=\log(1-z)$ for $\alpha=\frac{1}{2}$.

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